TOPICS IN SET THEORY: Example Sheet 2 [1]

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1 Special Aronszajn Trees

Prove that there exists a special Aronszajn tree.

2 $\mathbb{R}$-embeddable Trees

Suppose that $\mathbb{T}$ is an $\mathbb{R}$-embeddable normal $\aleph_1$-tree. Show that $\mathbb{T}$ is an Aronszajn tree, but not a Suslin tree.

3 Sub-trees

A tree $\mathbb{S}$ is a *convex sub-tree* of a tree $\mathbb{T}$ if $S \subseteq T$, $<_{\mathbb{S}} = <_{\mathbb{T}} \cap (S \times S)$, and $(\forall x \in S)(\forall y \in T)(y <_{\mathbb{T}} x \Rightarrow y \in S)$. Note that $ht_{<_{\mathbb{S}}}(x) = ht_{<_{\mathbb{T}}}(x)$ for every $x \in S$. A κ*-sub-tree* is a sub-tree that is also a κ-tree.

For $\delta \in Ord$ and a linear order I, let ${}^{<\delta}I$ be the family of I−valued sequences of length less than δ. Partially order ${}^{<\delta}I$ by $a \prec b$ iff b properly extends a, i.e. $a \subset b$. Show ${}^{<\delta}\mathbb{I} = ({}^{<\delta}I, \prec)$ is a tree. What are $ht({}^{<\delta}\mathbb{I})$ and $Lev_\alpha({}^{<\delta}\mathbb{I})$ for $\alpha < \delta$?

The tree ${}^{<\delta}\mathbb{I}$ is called the *complete (or full)* I-ary tree of height δ.

Prove that if the Aronszajn tree $\mathbb{A}$ is a convex sub-tree of $\{f \in {}^{<\omega_1}\omega : f$ is an injective function$\}$, then $\mathbb{A}$ is not a Suslin tree.

REMARK: Some sources (e.g. Kunen) require convexity in the definition of sub-tree; others do not. If convexity is not present, then $ht_{<_{\mathbb{S}}}(x) \leq ht_{<_{\mathbb{T}}}(x)$ and strict inequality can occur.

4 Singular Cardinals Lack the Tree Property

Show that no singular cardinal $\kappa \geq \aleph_0$ has the tree property, i.e. if κ is singular, then there exists a κ-Aronszajn tree.

5 Normalising Trees

(i) Show that if $\mathbb{T}$ is a normal $\aleph_1$-tree satisfying the countable chain condition, then $\mathbb{T}$ is a Suslin tree.

(ii) Prove that if $\mathbb{T}$ is a Suslin tree, then there are at most countably many $x \in T$ such that $\{y \in T : x <_{\mathbb{T}} y\}$ is countable.

(iii) Suppose that $\mathbb{S}$ is a Suslin tree. Explain (with proof) how to find inside $\mathbb{S}$ a normal Suslin tree $\mathbb{S}^*$ that satisfies the condition:

(*) $(\forall x \in S^*)(\{y \in S^* : y$ is an immediate $<_{\mathbb{S}^*}$-successor of $x\} = \aleph_0)$.

[1] Comments, improvements and corrections will be much appreciated; please send to ok261@cam.ac.uk; rev. 14/12/2014.

(iv) Suppose that $\mathbb{A}$ is an Aronszajn tree. Can one find inside $\mathbb{A}$ a normal Aronszajn tree $\mathbb{A}^*$ that satisfies the condition:

(*) $(\forall x \in A^*)(\{y \in A^* : y \text{ is an immediate } <_{\mathbb{A}^*}\text{-successor of } x\} = \aleph_0)$?

6 Dense Open Sets in Suslin Trees Are Co-Countable

Suppose $\mathbb{S}$ be a Suslin tree and $D \subseteq S$ is dense and open in $\mathbb{S}$: $(\forall x \in S)(\exists d \in D)(x \leq_{\mathbb{S}} d)$ and $(\forall z \in S)(\forall d \in D)(d <_{\mathbb{S}} z \Rightarrow z \in D)$. Prove D is co-countable in S.

7 Trees and Lines

Suppose $\mathbb{T}$ is a tree. For $\alpha < ht(\mathbb{T})$, let $\prec_\alpha$ be a linear ordering of the level T_α. For $x \in T$ and $\alpha \leq ht_{\mathbb{T}}(x)$, let $p_\alpha(x)$ be the unique element in $T_{\leq x} \cap T_\alpha$. Define a binary relation $\prec_{lex}$ on T as follows: for $x \neq y \in T$, let $x \prec_{lex} y$ iff either $x <_{\mathbb{T}} y$ or $p_\alpha(x) \prec_\alpha p_\alpha(y)$, where α is the least ordinal such that $p_\alpha(x) \neq p_\alpha(y)$. Show that $(T, \prec_{lex})$ is a linearly ordered set.

8 Aronszajn Lines and Specker Types

An *Aronszajn line* (also called a *Specker type*) is a linear order $\mathbb{A} = (A, <_{\mathbb{A}})$ such that A has cardinality $\aleph_1$, $\mathbb{A}$ contains no order-isomorphic copy of ω_1 or the reverse order-type ω_1^*, and $\mathbb{A}$ contains no order-isomorphic copy of an uncountable set of real numbers. Prove (in ordinary set theory) that Aronszajn lines exist.

9 κ-Kurepa Trees When $\kappa = \aleph_0$ or κ is (Strongly) Inaccessible

Suppose $\kappa = \aleph_0$ or κ is strongly inaccessible (i.e. $\kappa = cf(\kappa) > \aleph_0$ and $(\forall \lambda < \kappa)(2^\lambda < \kappa)$). Check that the complete binary tree of height κ is a κ–tree with at least κ^+ κ-branches.

10 Products of Trees

Suppose that for $i = 1, 2$, $\mathbb{T}_i$ is a κ–tree and λ_i, κ are infinite cardinals. Let the product tree $\mathbb{T}_1 \times \mathbb{T}_2 = (\bigcup_{\alpha<\kappa}(Lev_\alpha(T_1) \times Lev_\alpha(T_2)), \prec_{\mathbb{T}_1 \times \mathbb{T}_2})$ be defined as follows: $\langle x, y \rangle \prec_{\mathbb{T}_1 \times \mathbb{T}_2} \langle u, v \rangle$ iff $x \prec_{\mathbb{T}_1} u$ and $y \prec_{\mathbb{T}_2} v$.

(i) Verify that $\mathbb{T}_1 \times \mathbb{T}_2$ is a κ–tree.

(ii) Show that if $\mathbb{T}_1$ and $\mathbb{T}_2$ are Aronszajn trees, then $\mathbb{T}_1 \times \mathbb{T}_2$ is also an Aronszajn tree.

(iii) Prove that if $\mathbb{S}$ is a Suslin tree, then $\mathbb{S} \times \mathbb{S}$ is not a Suslin tree.

(iv) Find the optimal value μ (if it exists) as a function of λ_i such that if $\mathbb{T}_i$ is or has the property λ_i–P (and this makes sense), $i = 1, 2$, then the product $\mathbb{T}_1 \times \mathbb{T}_2$ is or has the property μ–P, where the property is listed below: (a) the μ–chain condition; (b) μ–Suslin; (c) μ–Aronszajn; (d) μ–Kurepa.

11 Suslin Trees from Suslin Lines

Suppose that there exists a Suslin line. Show there exists a Suslin tree.

12 Suslin Lines from Linear Orders $\mathbb{X}$ where $S(\mathbb{X}) \leq \aleph_1$ and $d(\mathbb{X}) = \aleph_1$

Suppose that there exists a linear order $\mathbb{X} = (X, <)$ such that $S(\mathbb{X}) \leq \aleph_1$ and $d(\mathbb{X}) = \aleph_1$.

(i) Show that there exists an unbounded dense linear order $\mathbb{Y} = (Y, <)$ such that $S(\mathbb{Y}) \leq \aleph_1$ and $d(\mathbb{Y}) = \aleph_1$.

(ii) Deduce there is a Suslin line.

13 Topological Properties of Suslin Lines

Suppose that $\mathbb{S}$ is a Suslin line. Prove the following claims.

(i) If $\{x_\alpha : \alpha < \eta\}$ is a strictly increasing sequence in $\mathbb{S}$, then $\eta < \omega_1$.

(ii) The line $\mathbb{S}$ contains a dense subset D of cardinality $\aleph_1$.

(iii) For each $x \in S$, there exists a strictly increasing sequence $\{x_n \in D : n < \omega\}$ such that $x = sup\{x_n \in D : n < \omega\}$.

(iv) The line S has cardinality $2^{\aleph_0}$.

14 Density Bounds

Suppose that $\mathbb{X} = (X, <)$ is a linear order such that $S(\mathbb{X}) \leq \aleph_1$. Prove $d(\mathbb{X}) \leq \aleph_1$

15 Aronszajn Tree Construction

(i) Show that there exists a family $\{e_\alpha : \alpha < \omega_1\}$ of functions such that each $e_\alpha : \alpha \to \omega$ is injective and for $\beta < \alpha, e_\beta$ and $e_\alpha \restriction \beta$ are identical at all but finitely many points.[2] [HINT. Define the functions by induction on $\alpha < \omega_1$; at successor ordinals, chose a one-point extension; at a limit ordinal δ, write $\delta = \sup_{n<\omega}\alpha_n$ where $\alpha_0 < \alpha_1 < \dots$ and use the functions e_{α_n}.]

(ii) Prove there exists an Aronszajn tree. [HINT. Consider the tree $\mathbb{T} = \{e_\alpha \restriction \beta : \beta \leq \alpha < \omega_1\}$ with the functions from above partially ordered by $f \prec g$ iff g is a proper extension of f.]

16 Countryman Types

A *Countryman type* (or *Countryman line*) is the order-type of an ordered set $\mathbb{C} = (C, <_\mathbb{C})$ if $C \times C$ is the union of countably many chains under the partial order $(a,b) \preceq (c,d) \Leftrightarrow (a \leq_\mathbb{C} c \wedge b \leq_\mathbb{C} d)$. Using the functions from the previous question, letting $C = \{e_\alpha : \alpha < \omega_1\}$ and $\mathbb{C} = (C, \prec_{lex})$, show that the order-type of $\mathbb{C}$ is a Countryman type.[3]

17 Suppose that $\mathbb{L}$ is a κ-Aronszajn line or a κ-Suslin line, i.e. derived from the pertinent κ-cultivar. What can one say about the existence of bisectors of uncountable subsets of $\mathbb{L}$?

[2] These functions will be used later in the proof that Cohen real forcing adds a Suslin tree in the generic extension; see Example Sheet 4.

[3] The existence of a Countryman line is due to Shelah (1976); the proof above was found by Todorcevic.

18 QUESTIONS.

(i) Can one refute the assertion "$2^{\aleph_\omega} > \aleph_{\omega+1}$ *and* $\aleph_{\omega+1}$ has the tree property"?

(ii) Can one refute the assertion "the tree property holds simultaneously for all $\aleph_n, 2 \leq n < \omega$ *and* for $\aleph_{\omega+1}$"?

(iii) Is it consistent (relative to ordinary set theory or even ordinary set theory plus some large cardinals) that for each uncountable cardinal κ, there is no κ^+-Aronszajn tree?

Recent research on Aronszajn trees has also explored the structural properties of the class, for example, the questions whether there is a universal Aronszajn tree, i.e. one in which every other is isomorphically embedded, whether there is a finite basis for the class, and whether the class of Aronszajn trees has a system of invariants, i.e. a classification theorem. The following items are useful:

(i) J. Cummings, M. Foreman, *The tree property*, Advances in Mathematics, 133 (1998), 1–32.

(ii) J. Tatch Moore, *Structural analysis of Aronszajn trees.*

`http://www.math.cornell.edu/~justin/Ftp/structure_Atree.pdf`

(iii) S. Todorcevic, *Trees and linearly ordered sets*, Handbook of Set-theoretic Topology, North-Holland 1984, 235-293.

(iv) B. Velickovic, *Saturation of Aronszajn trees and Shelah's conjecture.*

`http://www.logique.jussieu.fr/~boban/pdf/ucla-lecture.pdf`